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Wedderburn Specters and the Structure of Certain Associative Algebras*

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1. INTRODUCTION AND PRELIMINARIES

Let A be a finite-dimensional algebra over a field k with Jacobson radical J. In case A/J is a separable k-algebra, the Wedderburn Principal Theorem yields the k-space decomposition $A \simeq B \oplus J$ where B is a separable k-subalgebra of A and is uniquely determined up to inner automorphisms of A. In the general case one might hope for a separable subalgebra B which "becomes" a Wedderburn factor, i.e., a subalgebra B such that $B \otimes_k \bar{k}$ is a Wedderburn factor for $A \otimes_k \bar{k}$ over $\bar{k}$ (here $\bar{k}$ is an algebraic closure of k). In this paper we describe such potential Wedderburn factors and the algebras in which they occur.

If A/J is a separable k-algebra, the Wedderburn factors of A are just its maximal separable subalgebras. In general, however, maximal separable subalgebras need not stay maximal under base extension and do not necessarily have a uniquely determined k-dimension (see Example 2.6), so they cannot all be potential Wedderburn factors. We narrow our consideration to those separable subalgebras B over which A is purely inseparable.

DEFINITION 1.1. Let $R \supset S$ be rings. R is *purely inseparable* (usually abbreviated to PI) over S iff the R–R-bimodule map $\mu_{R/S}: R \otimes_S R^0 \to R$ determined by $\mu_{R/S}(r_1 \otimes_S r_2{}^0) = r_1 r_2$ has small kernel. (We will write just μ_R or μ for $\mu_{R/S}$ when we can do so without ambiguity.) In case S is commutative and R is an S-algebra, 1.1 reduces to the definition of a purely inseparable algebra extension given in [7]. We have a similar definition of separability.

DEFINITION 1.2. Let R, S, and μ be as in 1.1. R is *separable* over S iff μ splits. If R is an S-algebra, 1.2 reduces to the usual definition of separable algebra (see, e.g., [3]). The main result of this paper is the following theorem.

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THEOREM 2.16. *Let A be a finite-dimensional k-algebra with center C. Let $J = J(A)$, $\bar{A} = A/J$, and let $\pi\colon A \to \bar{A}$ be the natural surjection. Let L (resp. Λ) be the unique maximal separable k-subalgebra of C (resp. $Z(\bar{A})$). The following statements are equivalent for a separable k-subalgebra B of A.*

(1) *A is PI over B.*

(2) *If M is any field extension of k such that $(A \otimes_k M)/J(A \otimes_k M)$ is a separable M-algebra, then $B \otimes_k M$ is a Wedderburn factor for $A \otimes_k M$ over M.*

(3) *$\pi(B) \supset \Lambda$ and $\pi(Z_A(B)) \supset Z(\bar{A})$.*

(4) *$Z(B) \cong \Lambda$ via π and B is a maximal separable subalgebra of A.*

If A is semisimple, each of the above is equivalent to

(5) *$B \supset L$ and $A \cong B \otimes_L C$.*

Subalgebras B satisfying these properties are called *Wedderburn specters*. We give examples to show that Wedderburn specters need not exist, and need not be unique up to isomorphism when they do exist; however, for any finite-dimensional k-algebra A, $M_n(A)$ has a specter for some n.

Throughout this paper k is a field and A is a k-algebra. All rings and algebras are associative with unit. Subrings have the same unit as the over ring; all modules are unitary. For any ring R, R^0 denotes the opposite ring to R and $J(R)$ the Jacobson radical of R. $Z(R)$ is the center of R. If S is a subset of R, $Z_R(S) = \{r \in R \mid rs = sr \text{ for all } s \in S\}$.

If S is a subring of R, we form the tensor product $R \otimes_S R^0$ with slip-by $r_1 s \otimes_S r_2{}^0 = r_1 \otimes_S (sr_2)^0$. For any subring $C \subset S \cap Z(R)$, $R \otimes_S R^0$ and R are left $R \otimes_C R^0$-modules. The following fact is easily shown.

1.3. $\mu_{R/S}$ has small kernel (splits) as an R–R bimodule map iff it has small kernel (splits) as a map of left $R \otimes_C R^0$-modules.

1.3 allows us to rephrase the definitions of separability and pure inseparability in terms of the ambient ring $R \otimes_C R^0$ for any $C \subset S \cap Z(R)$. For convenience we note one more fact.

1.4. If R is both separable and PI over S, then μ induces an isomorphism $R \otimes_S R^0 \xrightarrow{\cong} R$ of left $R \otimes_C R^0$-modules (equivalently, of R–R bimodules). In fact any module map $f\colon M \to N$ which both splits and has small kernel induces an isomorphism.

2. PURE INSEPARABILITY AND WEDDERBURN SPECTERS

In case $A \supset B \supset k$ are all fields and B is separable over k, B is the maximal separable field extension of k in A iff A is purely inseparable over B. One implication holds more generally.

PROPOSITION 2.1. *Let $A \supset B$ be k-algebras. Suppose B is a separable k-algebra and A is* PI *over B. Then B is a maximal separable subalgebra of A over k.*

Proof. Suppose $A \supset S \supset B$ and S is a separable k-algebra. Let $\pi: A \otimes_B A^0 \to A \otimes_S A^0$ be the canonical map. The diagram below of left $A \otimes_k A^0$-modules commutes.

$$\begin{tikzcd} A \otimes_B A^0 \arrow[rr, "\mu_{A/B}"] \arrow[dr, "\pi"'] & & A \\ & A \otimes_S A^0 \arrow[ur, "\mu_{A/S}"'] & \end{tikzcd}$$

We claim π is an isomorphism. π has small kernel because $\ker \pi \subset \ker \mu_{A/B}$. By 1.4, to show π is an isomorphism, it will suffice to show it splits. Let $\tilde{\sigma} \in S \otimes_k S^0$ be a separability idempotent for S over k; that is, $\tilde{\sigma}$ defines a splitting for $\mu_{S/k}$ by $s \to (s \otimes_k 1^0) \cdot \tilde{\sigma}$. Let $\sigma = \operatorname{im}(\tilde{\sigma}) \in A \otimes_B A^0$ under the natural map $S \otimes_k S^0 \to A \otimes_B A^0$. Define $\rho: A \otimes_S A^0 \to A \otimes_B A^0$ by $\rho(\sum_i a_i \otimes_S \bar{a}_i^0) = (\sum_i a_i \otimes_k \bar{a}_i^0) \cdot \sigma$. ρ is the desired splitting for π.

We now have $A \otimes_B S \otimes_S A^0 \simeq A \otimes_B A^0 \simeq A \otimes_S A^0$, the composite map being induced by mult: $A \otimes_B S \to A$ where $\operatorname{mult}(a \otimes_B s) = as$. Since A^0 is faithfully flat over S, $A \otimes_B S \simeq A$.

The injection $S \to A$ induces an injection $S \otimes_B S \to A \otimes_B S$ because S is a flat B-module. mult | $S \otimes_B S$ yields an isomorphism $S \otimes_B S \xrightarrow{\approx} S$. B is separable over k, hence $B \otimes_k B^0$ is semisimple and we can write $S \simeq B \oplus S'$ as left $B \otimes_k B^0$-modules.

$$S \otimes_B S \simeq B \oplus (S' \otimes_B B) \oplus (B \otimes_B S') \oplus (S' \otimes_B S') \xrightarrow{\approx} B \oplus S'.$$

Let $s' \in S'$. Then $\operatorname{mult}(s' \otimes_B 1 - 1 \otimes_B s') = 0$ so

$$s' \otimes_B 1 = 1 \otimes_B s' \in (S' \otimes_B B) \cap (B \otimes_B S') = 0.$$

Hence $S' = 0$ and $B = S$. Q.E.D.

The converse of 2.1 need not hold. We will exhibit an algebra which has both "good" and "bad" maximal separable subalgebras; that is we find a subalgebra over which it is PI and another over which it is not. We need two lemmas.

LEMMA 2.2. *Let A be a finite-dimensional k-algebra with subalgebra B. Let R be a k-algebra such that $A \otimes_k A^0 \otimes_k R^0$ is an Artinian ring. Then A is* PI *over B iff $A \otimes_k R$ is* PI *over $B \otimes_k R$.*

Proof. Since k is a field the natural maps $A \to A \otimes_k R$ and $A \otimes_B A^0 \to$

$A \otimes_B A^0 \otimes_k R^0 (\cong (A \otimes_k R) \otimes_{B \otimes_k R} (A \otimes_k R)^0)$ are injections, and the diagram below commutes,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \mu_{A \otimes_k R} & \longrightarrow & A \otimes_B A^0 \otimes_k R^0 & \xrightarrow{\mu_{A \otimes_k R}} & A \otimes_k A & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker \mu_A & \longrightarrow & A \otimes_B A^0 & \xrightarrow{\mu_A} & A & \longrightarrow & 0
\end{array}$$

so the map $\ker \mu_A \to \ker \mu_{A \otimes_k R}$ is also an injection.

We claim $\ker \mu_{A \otimes_k R} = (\ker \mu_A) \otimes_k R^0$. Clearly $\ker \mu_{A \otimes_k R} \supseteq (\ker \mu_A) \otimes_k R^0$. Suppose $\alpha \in \ker \mu_{A \otimes_k R}$ and $\{r_i\}_{i \in I}$ is a k-basis for R. $\alpha = \sum_i \alpha_i \otimes_k r_i$ for some $\alpha_i \in A \otimes_B A^0$. $\mu_{A \otimes_k R}(\alpha) = 0$ implies $\mu_A(\alpha_i) = 0$ for all i. This completes the proof of the claim.

Suppose $(\ker \mu_A) \otimes_k R^0$ is small and $N + \ker \mu_A = A \otimes_B A^0$ for some submodule N of $A \otimes_B A^0$. We must show $N = A \otimes_B A^0$. $N \otimes_k R^0 + (\ker \mu_A \otimes_k R^0) = A \otimes_B A^0 \otimes_k R^0$; hence $N \otimes_k R^0 = A \otimes_B A^0 \otimes_k R^0$. Let $q: N \to A \otimes_B A^0$ be the natural injection. Since R^0 is a flat k-module, we get the exact sequence

$$0 \longrightarrow N \otimes_k R^0 \xrightarrow{q \otimes 1_{R^0}} A \otimes_B A^0 \otimes_k R^0 \longrightarrow ((A \otimes_B A^0)/N) \otimes_k R^0 \longrightarrow 0.$$

But $q \otimes 1_{R^0}$ is an isomorphism, so the last term is zero. Hence $(A \otimes_B A^0)/N = 0$ and $N = A \otimes_B A^0$.

Conversely, suppose $\ker \mu_A$ is small. For an Artinian ring S with left module M, it is well known that $J(S) \cdot M$ is the unique maximal small submodule of M (see, e.g., [2, pp. 120, 175]). We have $\ker \mu_A \subset J(A \otimes_k A^0) \cdot (A \otimes_B A^0)$ and we must show $(\ker \mu_A) \otimes_k R^0 \subset J(A \otimes_k A^0 \otimes_k R^0) \cdot (A \otimes_B A^0 \otimes_k R^0)$. It would suffice to know $J(A \otimes_k A^0) \otimes_k R^0 \subset J(A \otimes_k A^0 \otimes_k R^0)$. But $J(A \otimes_k A^0) \otimes_k R^0$ is a nilpotent ideal, hence is in the radical of $A \otimes_k A^0 \otimes_k R^0$. Q.E.D.

We state two special cases of the lemma for future reference.

COROLLARY 2.3. *Let $A \supset B \supset k$ be as in the lemma and let n be a positive integer. A is* PI *over B iff $M_n(A)$ is* PI *over $M_n(B)$.*

COROLLARY 2.4. *Let $A \supset B \supset k$ be as in the lemma. Let L be a field extension of k. A is* PI *over B iff $A \otimes_k L$ is* PI *over $B \otimes_k L$.*

LEMMA 2.5. *Let $R \supset S \supset T$ be rings.*

(a) *If R is* PI *over T, then R is* PI *over S.*

(b) *If R is separable over T, then R is separable over S.*

Proof. (a) $\ker \mu_{R/S} \subset \ker \mu_{R/T}$, so the former is small if the latter is.

(b) Let $\pi: R \otimes_T R^0 \to R \otimes_S R^0$ be the natural map. Suppose $\mu_{R/T}$ has separability idempotent σ. Then $\pi(\sigma)$ is a separability idempotent for $\mu_{R/S}$. Q.E.D.

EXAMPLE 2.6. Let $k = \mathbb{F}_2(\alpha)$, $C = \mathbb{F}_2(\alpha^{1/2})$, where $\mathbb{F}_2$ is the field of two elements and α is indeterminate over it. Let D be the C-algebra generated by x and y with relations

$$x^2 + x = 1,$$
$$y^2 = \alpha^{1/2},$$
$$xy + yx = y.$$

By [1, Theorem IX.26] D is a central quaternion division algebra over C. We will show that $M_2(k(x))$ is a maximal separable subalgebra of $M_2(D)$ over which it is not PI, and that $M_2(D)$ has another separable subalgebra (necessarily maximal by 2.1) over which it is PI.

$Z(M_2(k(x))) = k(x)$, a separable k-algebra, hence $M_2(k(x))$ is separable over k. $Z(M_2(D)) = C$, not a separable k-algebra, so $M_2(D)$ is not separable over k. (For characterizations of separable algebras see [3].) Now suppose S' is a k-algebra satisfying $M_2(D) \supsetneq S' \supsetneq M_2(k(x))$. Then $S' = M_2(S)$ for some k-algebra S with $D \supsetneq S \supsetneq k(x)$. Necessarily $S = k(x)(d)$ where $d \in D$ satisfies a quadratic equation over $k(x)$. Suppose $d^2 + (f + gx)d \in k(x)$ for $f, g \in k$. By examining coefficients in k of $\alpha^{1/2}$, $\alpha^{1/2}x$, y, $\alpha^{1/2}y$, xy, and $\alpha^{1/2}xy$ it can be shown that $d \in C(x)$, hence $S = C(x)$ and $S' = M_2(C(x))$, which is not separable over k. Therefore $M_2(k(x))$ is maximal separable.

By 2.5(b) D is separable over $C(x)$. If $M_2(D)$ were PI over $M_2(k(x))$, by 2.3 D would be PI over $k(x)$, and so by 2.5(a) D would be PI over $C(x)$. But then by 1.4 $D \otimes_{C(x)} D^0 \cong D$, which is false (count dimensions over the field $C(x)$). Thus $M_2(D)$ is not PI over $M_2(k(x))$.

We now describe a maximal separable k-subalgebra over which $M_2(D)$ is PI. By [6, Theorem 5] the map of Brauer groups $B(k) \to B(C)$ induced by $- \otimes_k C$ is surjective, so for some n there exists a central simple k-algebra E satisfying $M_n(D) \cong E \otimes_k C$. From the proof of [6, Theorem 5] we have $M_2(D) \cong E \otimes_k C$ where E is the following crossed product.

Let $Y = k(w)$ where $w^4 + w = 1$. Let $\tilde{x} = w^2 + w$. Let σ generate the automorphisms of Y leaving k fixed, e.g., take $\sigma(w) = w + \tilde{x}$. As Y-vector space E has basis $\{1, v, v^2, v^3\}$. Multiplication is determined by the equations $v^4 = \alpha$ and for all $y, z \in Y$, i and j integers,

$$yv^i zv^j = y\sigma^i(z)v^{i+j}.$$

To get an explicit isomorphism $\tau: M_2(D) \to E \otimes_k C$, note that E is a right D-module via the k-algebra injection $g: D \to E$ determined by

$$g(\alpha^{1/2}) = v^2,$$
$$g(x) = \tilde{x},$$
$$g(y) = v,$$

and $Z_E(g(D)) \cong C$. Hence $M_2(D) \cong \operatorname{End} E_D \cong E \otimes_k C$.

We compute τ using the D-basis $\{1, w\}$ of E.

$$\begin{aligned}
\tau(e_{11}) &= (w + 1) \otimes 1 + (w/\alpha)v^2 \otimes \alpha^{1/2},\\
\tau(e_{12}) &= 1 \otimes 1 + (1/\alpha)v^2 \otimes \alpha^{1/2},\\
\tau(e_{21}) &= \tilde{x} \otimes 1 + ((w + \tilde{x})/\alpha)v^2 \otimes \alpha^{1/2},\\
\tau(e_{22}) &= w \otimes 1 + (w/\alpha)v^2 \otimes \alpha^{1/2},\\
\tau(x) &= \tilde{x} \otimes 1,\\
\tau(y) &= \tilde{x}v \otimes 1,\\
\tau(xe_{12} + e_{21} + e_{22}) &= w \otimes 1,\\
\tau((xy + y)\, e_{12} + xy) &= v \otimes 1.
\end{aligned}$$

From now on we identify x and $\tilde{x}$.

E is a central simple k-algebra, hence separable. Let $(\ )^0\colon E^0 \to E$ be the map $(e^0)^0 = e$. The diagram commutes,

$$\begin{tikzcd}
(E \otimes_k C) \otimes_E (E \otimes_k C)^0 \arrow[r, "\mu_{E \otimes_k C}"] \arrow[d, "\wr" , no head, equal] & E \otimes_k C \arrow[d, dotted, no head] \\
E^0 \otimes_k C \otimes_k C \arrow[r, "(\ )^0 \otimes_k \mu_C"] & E \otimes_k C
\end{tikzcd}$$

so $\ker \mu_{E \otimes_k C} \cong E^0 \otimes_k \ker \mu_C$. Since C is PI over k, $\ker \mu_C \subset J(C \otimes_k C)$ is a nilpotent ideal. $E^0 \otimes_k \ker \mu_C$ is therefore also a nilpotent ideal; i.e., a small submodule of $E^0 \otimes_k C \otimes_k C$. We have shown $M_2(D)$ is PI over E. By 2.1, E must be a maximal separable subalgebra of $M_2(D)$.

The next lemma characterizes maximal separable subalgebras in an algebra R for which $R/J(R)$ is separable.

LEMMA 2.7. *Let R be a finite-dimensional k-algebra with $R/J(R)$ separable. Let S be a separable subalgebra of R. The following statements are equivalent.*

(a) *S is a Wedderburn factor for R over k.*

(b) *R is* PI *over S.*

(c) *S is a maximal separable subalgebra of R.*

Proof. (a) $\to$ (b). Let $J = J(R)$. $R \simeq S \oplus J$ as $S \otimes_k S^0$-modules and the diagram commutes.

$$\begin{tikzcd}
R \otimes_S R^0 \arrow[r, "\mu_R"] \arrow[d, "\wr", no head, equal] & R \arrow[d, "\wr", no head, equal] \\
S^0 \oplus (J \otimes_S S^0) \oplus (S \otimes_S J^0) \oplus (J \otimes_S J^0) \arrow[r, "(\ )^0 \oplus \mathrm{mult} \oplus \mathrm{mult} \oplus \mathrm{mult}"] & S \oplus J
\end{tikzcd}$$

Since $(\ )^0$ is an isomorphism of $S \otimes_k S^0$-modules,

$$\begin{aligned}\ker \mu_R &\subset ((J \otimes_S R^0) \oplus (R \otimes_S J^0)\\ &\subset ((J \otimes_K R^0) + (R \otimes_k J^0)) \cdot (R \otimes_S R^0) \subset J(R \otimes_k R^0) \cdot (R \otimes_S R^0),\end{aligned}$$

the last inclusion holding because $(J \otimes_k R^0) + (R \otimes_k J^0)$ is a nilpotent ideal. Since $R \otimes_k R^0$ is Artinian, $J(R \otimes_k R^0) \cdot (R \otimes_S S^0)$ is small.

(b) $\rightarrow$ (c). This implication is a special case of 2.1.

(c) $\rightarrow$ (a). It would suffice to know that every separable subalgebra of R is contained in a Wedderburn factor, which can be shown by essentially the same argument used to prove uniqueness of Wedderburn factors in the Wedderburn–Mal'cev theorem in [4]. Q.E.D.

We are now ready to say something about maximal separable subalgebras in a more general setting.

PROPOSITION 2.8. *Let A be a finite-dimensional k-algebra with separable subalgebra B. Let L be any field extension of k such that $(A \otimes_k L)/J(A \otimes_k L)$ is a separable L-algebra. Then A is* PI *over B iff $B \otimes_k L$ is a Wedderburn factor for $A \otimes_k L$ over L.*

Proof. Since B is a separable k-algebra, $B \otimes_k L$ is a separable L-algebra. A is PI over B iff $A \otimes_k L$ is PI over $B \otimes_k L$ (by 2.4) iff $B \otimes_k L$ is a Wedderburn factor for $A \otimes_k L$ over L (by the preceding lemma). Q.E.D.

If one (hence both) of these conditions holds, B is called a *Wedderburn specter* (specter for short) *for A over k.*

COROLLARY 2.9. *Let A be a finite-dimensional algebra over a field k. All specters for A over k have the same dimension, which is strictly larger than the dimension of any separable subalgebra not a specter.*

Proof. Let B be a separable subalgebra of A. Let $\bar{k}$ be an algebraic closure of k. By the proposition B is a specter iff

$$[B : k] = [A \otimes_k \bar{k} : \bar{k}] - [J(A \otimes_k \bar{k}) : \bar{k}]. \qquad \text{Q.E.D.}$$

The next three propositions determine which algebras have specters.

PROPOSITION 2.10. *Let A be a finite-dimensional k-algebra. Let $J = J(A)$ and $\bar{A} = A/J$. Then A has a specter over k iff $\bar{A}$ does. If $\bar{A}$ has a unique specter and no other maximal separable subalgebras, then every maximal separable subalgebra of A is a specter and any two specters in A are isomorphic by an inner automorphism.*

Proof. Let $\pi: A \rightarrow \bar{A}$ be the natural map.

Suppose B is a specter for A. B is separable so $\pi \mid B$ is an injection. Let $\bar{B} = \pi(B)$. We will show that $\bar{B}$ is a specter for $\bar{A}$. $\bar{B} \simeq B$, hence $\bar{B}$ is separable. It remains to show that $\bar{A}$ is PI over $\bar{B}$. Below is a commutative diagram of $A \otimes_k A^0$-modules with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & A \otimes_B A^0 & \xrightarrow{\mu_A} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi\otimes\pi^0 \mid L} & & \downarrow{\scriptstyle \pi\otimes\pi^0} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & K & \longrightarrow & \bar{A} \otimes_{\bar{B}} \bar{A}^0 & \xrightarrow{\mu_{\bar{A}}} & \bar{A} & \longrightarrow & 0 \end{array} \tag{2.11}$$

Here $L = \ker \mu_A$ and $K = \ker \mu_{\bar{A}}$. If $\bar{\beta} \in K$, let β be some element in the inverse image $(\pi \otimes \pi^0)^{-1}(\bar{\beta})$. Then $b = \beta - (\mu_A(\beta) \otimes 1^0) \in L$ and $(\pi \otimes \pi^0)(b) = \bar{\beta}$, so $\pi \otimes \pi^0(L) = K$. K is small since L is.

Conversely, suppose $\bar{A}$ has a specter $\bar{B}$. Let $B' = \pi^{-1}(\bar{B})$. Then $B'/J(B') = B'/J \simeq \bar{B}$, a separable k-algebra. By the Wedderburn Principal Theorem, B' has a Wedderburn factor $B \simeq \bar{B}$. We will show that B is a specter for A.

B is a Wedderburn factor hence separable over k. It remains to show that L is small. From diagram (2.11) above

$$L = \ker \mu_A \subset \ker(\pi \circ \mu_A) = \ker(\mu_{\bar{A}} \circ (\pi \otimes \pi^0)).$$

The kernel of a composition of maps is small if each factor map has small kernel, so to complete the proof that B is a specter it will suffice to show $\ker(\pi \otimes \pi^0)$ is small.

Since B is separable over k $B \otimes_k B^0$ is semisimple. As a $B \otimes_k B^0$-submodule of A, J has a complement A'.

$$A \otimes_B A^0 \simeq (A' \otimes_B (A')^0) \oplus (A' \otimes_B J^0) \oplus (J \otimes_B (A')^0) \oplus (J \otimes_B J^0).$$

$\pi \otimes \pi^0$ is an isomorphism on the first summand, so

$$\begin{aligned} \ker(\pi \otimes \pi^0) &\subset (A' \otimes_B J^0) + J \otimes_B (A')^0) \\ &\subset (A \otimes_B J^0) + (J \otimes_B A^0) \\ &\subset ((A \otimes_k J^0) + (J \otimes_k A^0)) \cdot (A \otimes_B A^0) \\ &\subset J(A \otimes_k A^0) \cdot (A \otimes_B A^0), \end{aligned}$$

which is small since $A \otimes_k A^0$ is Artinian (cf. the proof of 2.2).

Now suppose that $\bar{A}$ has a unique maximal separable subalgebra $\bar{B}$, and that $\bar{B}$ is a specter. Suppose B and $\tilde{B}$ are specters for A over k. By the first part of the proof, $\pi(B) = \pi(\tilde{B}) = \bar{B}$, so B and $\tilde{B}$ are Wedderburn factors for $\pi^{-1}(\bar{B})$. By the Wedderburn–Mal'cev theorem, $B \simeq \tilde{B}$ by an inner automorphism of $\pi^{-1}(\bar{B}) \subset A$. Q.E.D.

PROPOSITION 2.12. *Let $A_i \supset B_i$ be finite-dimensional k-algebras for $1 \leq i \leq n$. Then $\oplus_i B_i$ is a specter for $\oplus_i A_i$ iff B_i is a specter for A_i for $1 \leq i \leq n$.*

Proof. By [1, p. 44] $\oplus_i B_i$ is separable over k iff each B_i is separable over k. $\ker \mu_{\oplus A_i} \simeq \oplus_i \ker \mu_{A_i}$, and by well-known properties of small submodules (see, e.g., [2]) $\oplus_i \ker \mu_{A_i}$ is small iff $\ker \mu_{A_i}$ is small for all i. Q.E.D.

COROLLARY 2.13. *Let A be a finite-dimensional solvable k-algebra (i.e., $A/J(A)$ is commutative). Then A has a specter unique up to inner automorphism and every maximal separable subalgebra is a specter. In particular any finite-dimensional commutative k-algebra has a specter which is its unique maximal separable subalgebra.*

Proof. By the two preceding propositions it will suffice to show that every finite-dimensional simple commutative k-algebra (i.e., field extension) R has a unique maximal separable subalgebra S and that R is PI over S. Take $S =$ separable closure of k in R. Q.E.D.

PROPOSITION 2.14. *Let A be a simple finite-dimensional k-algebra. Let $C = Z(A)$ and $L =$ separable closure of k in C. Then A has a Wedderburn specter iff there exists a central simple L-algebra E such that $A \simeq E \otimes_L C$. In this case E is a specter for A over k.*

Proof. Suppose $A \simeq E \otimes_L C$. By the same argument used in Example 2.6, E is a specter for A over L. But E is separable over k since its center is, so E is also a specter for A over k. We have incidentally proved the final statement in the proposition.

We first prove the converse for the special case $L = k$. We assume A is a central simple C-algebra and C is a PI field extension of k. Let B be a specter for A over k. We will show $A \simeq B \otimes_k C$. As a first step we show that $(A \otimes_k C)/J(A \otimes_k C)$ is a separable C-algebra.

Let $m: A \otimes_k C \to A$ be the algebra surjection determined by $m(a \otimes_k c) = ac$. To finish the first step we need only show that $J(A \otimes_k C) = \ker m$. Now $J(A \otimes_k C)$ is a nilpotent ideal, hence $m(J(A \otimes_k C))$ is also. But A is simple; its only nilpotent ideal is (0), so $J(A \otimes_k C) \subset \ker m$.

Suppose $\alpha \in \ker m$ and let $\{a_i\}$ be a C-basis for A. $\alpha = \sum_i (\sum_{l_i} a_i c_{l_i} \otimes_k c_{l_i}')$ for some $c_{l_i}, c_{l_i}' \in C$. $\sum_{i,l_i} a_i c_{l_i} c_{l_i}' = 0$ implies $\sum_{l_i} c_{l_i} c_{l_i}' = 0$ for all i, so

$$\sum_{i,l_i} c_{l_i} \otimes_k c_{l_i}' \in \ker \mu_C \subset J(C \otimes_k C).$$

Now $J(C \otimes_k C)$ is nilpotent and central in $A \otimes_k C$, hence it generates a nilpotent ideal I in $A \otimes_k C$. $\alpha \in I \subset J(A \otimes_k C)$. This completes the proof that $A \otimes_k C/J(A \otimes_k C) \simeq A$.

Now apply 2.8 to get $B \otimes_k C \simeq A \otimes_k C/J(A \otimes_k C) \simeq A$. Since A is simple, B is also.

$$C = Z(A) = Z(B) \otimes_k Z(C) = Z(B) \otimes_k C,$$

hence $Z(B) = k$.

Now we are ready for the proof of the general case. Let A, C, L, k be as in the statement of the proposition, and suppose B is a specter for A over k. By the special case, it would be enough to show B is also a specter for A over L. Since A is PI over B we only need to know that B is a separable L-algebra. By 2.5(b) it will suffice to show $B \supset L$.

Since B and L are separable k-algebras, $B \otimes_k L$ is also; hence the homomorphic image BL of $B \otimes_k L$ under the multiplication map $B \otimes_k L \to A$ is separable. $BL \supset B$, so by maximality $B = BL \supset L$. Q.E.D.

COROLLARY 2.15. *Let A be a finite-dimensional k-algebra. Then for some n, $M_n(A)$ has a specter over k.*

Proof. For each simple component A_i of $A/J(A)$ there is a positive integer n_i such that $M_{n_i}(A_i)$ has a specter (apply [6, Theorem 5] and the preceding proposition). Let n be the least common multiple of the n_i's. Then $M_n(A/J(A)) \cong M_n(A)/M_n(J(A))$ has a specter. By 2.10 $M_n(A)$ does also. Q.E.D.

THEOREM 2.16.[1] *Let A be a finite-dimensional k-algebra with center C. Let $J = J(A)$, $\bar{A} = A/J$, and let $\pi\colon A \to \bar{A}$ be the natural surjection. Let L (resp. Λ) be the unique maximal separable k-subalgebra of C (resp. $Z(\bar{A})$). The following statements are equivalent for a separable k-subalgebra B of A.*

(1) *A is PI over B (i.e., B is a specter for A over k).*

(2) *If M is any field extension of k such that $(A \otimes_k M)/J(A \otimes_k M)$ is a separable M-algebra, then $B \otimes_k M$ is a Wedderburn factor for $A \otimes_k M$ over M.*

(3) *$\pi(B) \supset \Lambda$ and $\pi(Z_A(B)) \subset Z(\bar{A})$.*

(4) *$Z(B) \cong \Lambda$ via π and B is a maximal separable subalgebra of A.*

If A is semisimple, each of these statements is equivalent to

(5) *$B \supset L$ and $A \cong B \otimes_L C$.*

Proof. (1) $\leftrightarrow$ (2) by 2.8. We prove (1) $\leftrightarrow$ (3) in three stages.

(i) Assume A is simple. In this case we restate (3) as

(3′) $B \supset L$ and $Z_A(B) = Z(A) = C$.

(1) $\to$ (3′). By the proof of 2.14 B is a central simple L-algebra and $A \cong B \otimes_L C$. By the standard theory of simple algebras (see, e.g., [5, Theorem 4.2]) $A \cong B \otimes_L Z_A(B)$. Since $C \subset Z_A(B)$, $B \otimes_L C \cong B \otimes_L Z_A(B)$ implies $C = Z_A(B)$.

[1] The author is indebted to E. C. Ingraham for pointing out a mistake in an earlier version of this theorem.

$(3') \to (1)$. $C = Z_A(B) \supset Z_A(B) \cap B = Z(B) \supset L$. Since B is a separable k-algebra, its center is also, hence $Z(B) = L$. B is simple because its center is a field. By [5, Theorem 4.2] $A \simeq B \otimes_L Z_A(B) \simeq B \otimes_L C$. By 2.14, B is a specter.

(ii) Assume A is semisimple. In this case also (3) is equivalent to (3'). Let $A = \bigoplus_{i=1}^n A_i$ with A_i simple, and suppose $B = \bigoplus_{i=1}^n B_i$, $L = \bigoplus_{i=1}^n L_i$, and $C = \bigoplus_{i=1}^n C_i$ where $C_i, L_i, B_i \subset A_i$ for $1 \leqslant i \leqslant n$. B is a specter for A iff each B_i is a specter for A_i (by 2.12) iff $Z_{A_i}(B_i) = C_i$ and $B_i \supset L_i$ (by the simple case) iff $Z_A(B) = C$ and $B \supset L$. The last equivalence holds because

$$Z_A(B_i) = A_1 \oplus \cdots \oplus A_{i-1} \oplus Z_{A_i}(B_i) \oplus \cdots \oplus A_n,$$

hence $Z_A(B) = Z_A(\bigoplus_i B_i) = \bigcap_i Z_A(B_i) = \bigoplus_i Z_{A_i}(B_i)$.

(iii) Assume A is a finite-dimensional k-algebra. Let $\bar{B} = \pi(B)$. $\pi \mid B$ is injective.

By the proof of 2.10 B is a specter for A iff $\bar{B}$ is a specter for $\bar{A}$, so by the semisimple case it will be enough to show the equivalence of (3) and

(3″) $\bar{B} \supset A$ and $Z_{\bar{A}}(\bar{B}) = Z(\bar{A})$.

$(3) \to (3'')$. It is enough to show $Z_{\bar{A}}(\bar{B}) \subset Z(\bar{A})$. Suppose $\bar{q} \in Z_{\bar{A}}(\bar{B})$. Let $q \in \pi^{-1}(\bar{q})$. Then $qb - bq \in J$ for all $b \in B$, i.e., q induces a derivation from B to J. Since B is separable, all derivations are inner [4, Theorem 72.16] so there exists $n \in J$ with $nb - bn = qb - bq$ for all $b \in B$ and $q - n \in Z_A(B)$. Hence $\bar{q} = \pi(q - n) \in \pi(Z_A(B)) \subset Z(\bar{A})$, where the last inclusion comes from (3).

$(3'') \to (3)$. Suppose $a \in Z_A(B)$. Then $\pi(a) \in Z_{\bar{A}}(\pi(B)) = Z_{\bar{A}}(\bar{B}) = Z(\bar{A})$ by (3″).

$(1) \to (4)$. By 2.1 B is maximal separable. $\pi \mid B$ is injective and $\bar{B}$ is a specter for $\bar{A}$ by 2.10, so $Z(\bar{B}) = \Lambda$. Hence $\pi\colon Z(B) \to \Lambda$ is an isomorphism.

$(4) \to (1)$. It is enough to show that each simple component of $\bar{A}$ has a specter which is its intersection with $\bar{B}$. The hypotheses of (4) on A and B imply the same hypotheses on the simple components of $\bar{A}$, so without loss of generality we may assume A is simple with maximal separable subalgebra B and $Z(B) = L$, a separable field extension of k.

Since B is semisimple with center a field, it is simple and $A \simeq B \otimes_L Z_A(B)$. $Z_A(B)$ is a simple ring. By 2.14 it will suffice to show $Z_A(B) = C$.

Suppose $a \in Z_A(B)$ satisfies a separable equation over L. Then the image of $B \otimes_L L[a]$ in A under multiplication is a separable k-subalgebra of A containing B. By maximality, $BL[a] = B$ so $a \in L$; i.e., every element of $Z_A(B)$ satisfying a separable equation over L is already in L. In particular, $Z_A(B)$ has no idempotents so it is a division algebra. By [5, Theorem 3.2.1] it is commutative. $C = Z(A) \simeq Z(B) \otimes_L Z(Z_A(B)) \simeq L \otimes_L Z_A(B)$, so $Z_A(B) = C$.

$(1) \leftrightarrow (5)$. In case A is simple, the equivalence is essentially a restatement of 2.14. Now let $A = \oplus_i A_i$, $B = \oplus_i B_i$, $C = \oplus_i C_i$, and $L = \oplus_i L_i$ where L_i, C_i, $B_i \subset A_i$ and A_i is simple for $1 \leqslant i \leqslant n$. B is a specter for A iff each B_i is a specter for A_i (by 2.12) iff $B_i \supset L_i$ and $A_i \simeq B_i \otimes_{L_i} C_i$ (by the simple case) iff $B \supset L$ and

$$A = \bigoplus_i A_i \simeq \bigoplus_i (B_i \otimes_{L_i} C_i) \simeq \left(\bigoplus_i B_i\right) \otimes_L \left(\bigoplus_l C_l\right)$$

since $B_i \otimes_{L_i} C_l = 0$ unless $i = j = l$. Q.E.D.

We conclude with examples of two kinds of problems which can arise in the study of specters.

EXAMPLE 2.17. Specters are not unique up to isomorphism.

Specters for a simple algebra A are characterized by their behavior under certain base extensions (see 2.8). Since in general the functor $-\otimes_k C$ from k-algebras to C-algebras is not one-to-one on isomorphism classes (here C is a field extension of k), one cannot expect specters to be unique up to isomorphism. For example, let $k = \mathbb{F}_2(\alpha)$, $C = \mathbb{F}_2(\alpha^{1/2})$ (as in 2.6) and $A = M_2(C)$. $M_2(C) \simeq M_2(k) \otimes_k C$, so $M_2(k)$ is a specter for $M_2(C)$ over k. Let $\underset{\sim}{Q}$ be quaternions over k with k-basis $\{1, \xi, \eta, \xi\eta\}$ and relations

$$\begin{aligned} \xi^2 + \xi &= 1, \\ \eta^2 &= \alpha, \\ \xi\eta + \eta\xi &= \eta. \end{aligned}$$

$\underset{\sim}{Q}$ injects into $M_2(C)$ by

$$\xi \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \eta \mapsto \begin{pmatrix} 0 & \alpha^{1/2} \\ \alpha^{1/2} & 0 \end{pmatrix},$$

so $\underset{\sim}{Q}$ is a separable k-subalgebra of $M_2(C)$ of the same dimension as $M_2(k)$. By 2.9 $\underset{\sim}{Q}$ is also a specter for $M_2(C)$. But $\underset{\sim}{Q} \not\simeq M_2(k)$ since $\underset{\sim}{Q}$ is a division algebra [1, Theorem IX.26].

EXAMPLE 2.18. Not all algebras have specters.

Let D, C, and k be as in 2.6; i.e., $k = \mathbb{F}_2(\alpha)$, $C = \mathbb{F}_2(\alpha^{1/2})$, and D a quaternion algebra over C generated by x and y with relations

$$\begin{aligned} x^2 + x &= 1, \\ y^2 &= \alpha^{1/2}, \\ xy + yx &= y. \end{aligned}$$

Suppose D had a specter G. Then $D \simeq G \otimes_k C$ and G would be a four-dimensional central simple k-algebra. Since D has no idempotents, G would have to be a division algebra. We claim any four-dimensional central division algebra

Q over k is split by C. For by [1, Lemma IX.8] Q is cyclic, hence by [1, Theorem VII.27] Q has a purely inseparable splitting field of degree two over k. To finish the proof of the claim it will suffice to prove the following lemma.

LEMMA 2.19. *The only second degree* PI *field extension of k is C.*

Proof. Let $k(\theta)$ be such an extension. $\theta^2 = f(\alpha)/g(\alpha)$ for some $f(\alpha), g(\alpha) \in \mathbb{F}_2[\alpha]$, and $k(\theta) = k(\theta g(\alpha))$, so without loss of generality assume $\theta^2 = f(\alpha) \in \mathbb{F}_2[\alpha]$. Write $\theta^2 = (f_1(\alpha))^2 + \alpha(f_2(\alpha))^2$ for some $f_1(\alpha)$, $f_2(\alpha) \in \mathbb{F}_2[\alpha]$. Then $k(\theta) = k((\theta + f_1(\alpha))/f_2(\alpha)) = k(\alpha^{1/2}) = C$. Q.E.D.

Hence if a specter G existed, we would have

$$D \cong G \otimes_k C \cong M_2(C),$$

a contradiction.

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